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A gap theorem for complete constant scalar curvature hypersurfaces in the de Sitter space

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Abstract

To each immersed complete space-like hypersurface M with constant normalized scalar curvature R in the de Sitter space S_1^{n+1}, we associate $\sup H^2$, where H is the mean curvature of M. It is proved that the condition $\sup H^2 \leq C_n(\bar{R})$, where $\bar{R} = (R-1) > 0$ and $C_n(\bar{R})$ is a constant depending only on R and n, implies that either M is totally umbilical or M is a hyperbolic cylinder. It is also proved the sharpness of this result by showing the existence of a class of new rotation constant scalar curvature hypersurfaces in S_1^{n+1} such that $\sup H^2 > C_n(\bar{R})$.

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1. Introduction

Let $\mathbf{R}_1^{n+2}$ be the real vector space $\mathbf{R}^{n+2}$ endowed with the Lorentzian metric $\langle , \rangle$ given by $\langle v, w\rangle = -v_0w_0 + v_1w_1 + \cdots + v_{n+1}w_{n+1}$; that is, $\mathbf{R}_1^{n+2} = L^{n+2}$ is the Lorentz–Minkowski $(n+2)$-dimensional space. We define the de Sitter space as the following hyperquadric of $\mathbf{R}_1^{n+2}$: $S_1^{n+1} = \{x \in \mathbf{R}_1^{n+2}; |x|^2 = 1\}$. The induced metric $\langle , \rangle$ makes S_1^{n+1} into a Lorentz manifold with constant sectional curvature 1.

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In 1977, Goddard [8] conjectured that the only complete space-like hypersurfaces (those whose induced metric is Riemannian) in S_1^{n+1} with constant mean curvature are the totally umbilical ones. Montiel [12] proved this conjecture in the compact case and exhibited a counterexample for the general case. The motivation to the study of space-like hypersurfaces in space–times comes from its relevance in general relativity. In particular, the de Sitter space is a space–time model with constant sectional curvature. On the other hand, constant mean curvature hypersurfaces are relevant for studying propagation of gravitational waves.

It is quite natural to pose a Goddard-like question for constant scalar curvature hypersurfaces in S_1^{n+1}. Partial results were obtained in [5,20,21]. For the constant scalar curvature compact Riemannian case in spheres, a rigidity theorem is given in [10]. Recently, Li [11] proved that a compact space-like hypersurface with constant normalized scalar curvature $R > 1$ in S_1^{n+1} must be totally umbilical (see also [14] for a similar result in a more general space–time and [1] for related results involving general r-mean curvatures). On the other hand, Montiel [13] proved that a complete space-like hypersurface with constant mean curvature $H^2 = 4(n-1)/n^2$ in S_1^{n+1} and more than one topological end is isometric to the hyperbolic cylinder $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$.

In this paper, we substitute Montiel hypotheses for constant normalized scalar curvature $R > 1$ and $\sup H^2 \le C_n(\bar{R})$, where H is the mean curvature and $C_n(\bar{R})$ is a constant depending only on the dimension $n \ge 3$ and $\bar{R} = R - 1$. We then classify all hypersurfaces satisfying these conditions. More precisely, we prove the following theorem (which was announced in [3]).

Theorem 1.1. *Let M^n be an n-dimensional $(n \ge 3)$ complete space-like hypersurface immersed in S_1^{n+1} with constant normalized scalar curvature R and $\bar{R} = R - 1 > 0$. Suppose also that* $\sup H^2 \le C_n(\bar{R})$, *where*

$$C_n(\bar{R}) = \frac{1}{n^2}\left((n-1)^2\frac{n\bar{R}-2}{n-2} + 2(n-1) + \frac{n-2}{n\bar{R}-2}\right). \tag{1}$$

Then either

1. $\sup H^2 = \bar{R}$ *and M is totally umbilical; or*
2. $\sup H^2 = C_n(\bar{R})$ *and M is isometric to the hyperbolic cylinder*

$$H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r).$$

Theorem 1.1 has the following consequences:

1. The only complete constant normalized scalar curvature space-like hypersurfaces with $\bar{R} = \sup H^2$ are the umbilical ones.
2. There are no complete constant normalized scalar curvature space-like hypersurfaces such that $\bar{R} < \sup H^2 < C_n(\bar{R})$.
3. The only complete constant normalized scalar curvature space-like hypersurfaces with $\sup H^2 = C_n(\bar{R})$ are the hyperbolic cylinders.

We also show that these characterizations are sharp, by proving the existence, for every number $C > C_n(\bar{R})$, of a complete constant normalized scalar curvature space-like hypersurface distinct from the above ones and such that $\sup H^2 = C$. These examples constitute a class of new rotation hypersurfaces with constant scalar curvature in S_1^{n+1}. In fact, we prove the following theorem.

Theorem 1.2. *Fix an integer $n \geq 3$ and let $\bar{R} \in (2/n, +\infty)$. Then, for each $C \geq C_n(\bar{R})$, there exists a complete immersed space-like hypersurface in S_1^{n+1} with normalized scalar curvature $R = \bar{R} + 1$ and mean curvature H satisfying* $\sup H^2 = C$. *Moreover, if $C = C_n(\bar{R})$, then the hypersurface is a hyperbolic cylinder.*

Chern et al. [4] associated to each compact hypersurface in the Euclidean sphere S^{n+1} with $H = 0$, the square of the norm of its second fundamental form (or equivalently, its scalar curvature) and asked if the image of such a function is discrete. A Lorentzian version dual of this question could be to associate to each complete space-like hypersurface in S_1^{n+1} with constant scalar curvature the value $\sup H^2$ and ask if the image of such a function is discrete.

The results we obtain give, for each $n \geq 3$, a detailed picture of the set G_n of the complete constant normalized scalar curvature space-like hypersurfaces in S_1^{n+1}. In fact, our results can be interpreted as follows: given $n \geq 3$, we associate to each such a complete constant normalized scalar curvature space-like hypersurface M^n the coordinate pair $(\bar{R}, \sup H^2)$ in the first quadrant of a 2-plane, thus obtaining Fig. 1. In this figure, the bisector ray

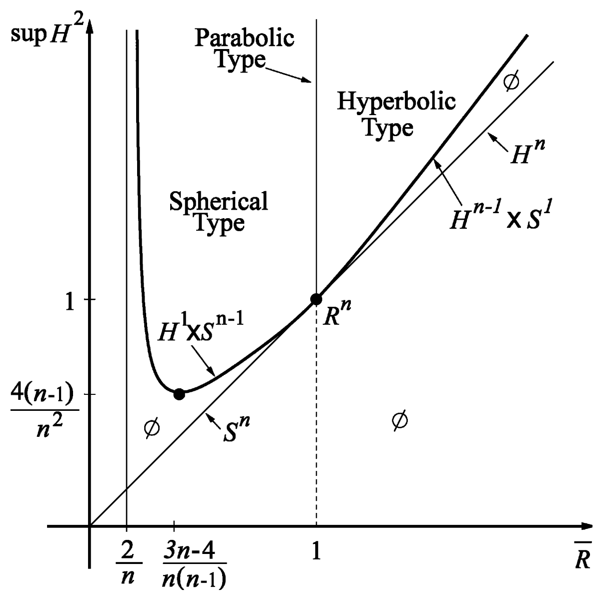

Fig. 1. The plane $(\bar{R}, \sup H^2)$, where the curve shown is the graph of $C_n(\bar{R})$ corresponding to the hyperbolic cylinders $H^1 \times S^{n-1}$. The region above the graph is associated to the rotation hypersurfaces given by Theorem 1.2, classified in three types by Definition 4.1. The bisector ray represents the umbilical hypersurfaces and, according to Theorem 1.1, regions marked with $\varnothing$ do not have any associated complete constant scalar curvature space-like hypersurface.

corresponds to the umbilical hypersurfaces and the curve shown there corresponds to the hyperbolic cylinders. We see that for each $\bar{R} \neq 1$, there is no complete hypersurface of the kind studied here and such that the corresponding point $(\bar{R}, \sup H^2)$ lies above the bisector ray and below the curve, thus yielding a gap (see Section 4). Our results also show that for n and $\bar{R}$ fixed, the image of the function $G_n \to \mathbf{R}$ given by $M^n \mapsto \sup H^2$ is the set $\{\bar{R}\} \cup [C(\bar{R}), \infty)$, which obviously is not discrete, thus giving an answer to a Lorentzian version dual to the Chern–do Carmo–Kobayashi's question.

2. Preliminaries

Let M^n be an n-dimensional complete orientable manifold and let $f : M^n \to S_1^{n+1} \subset \mathbf{R}_1^{n+2}$ be a space-like immersion of M^n into the $(n+1)$-dimensional de Sitter space S_1^{n+1}. Choose a unit normal η along f and denote by $A : T_pM \to T_pM$ the linear map of the tangent space T_pM at the point $p \in M$, associated to the second fundamental form of f along η,

$$\langle AX, Y\rangle = -\langle \nabla_X Y, \eta\rangle,$$

where X and Y are tangent vector fields on M and ∇ is the connection on S_1^{n+1}. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis which diagonalizes A with eigenvalues k_i, i.e., $Ae_i = k_i e_i$, $i = 1, \dots, n$. $|A|^2 = \sum k_i^2$ is the square of the norm of the second fundamental form, $H = (1/n)\sum k_i$ is the mean curvature of f and $R = \bar{R} + 1$ is the normalized scalar curvature, where

$$\bar{R} = \sum_{i \neq j} k_i k_j.$$

In our case it is convenient to define a linear map $\phi : T_pM \to T_pM$ by

$$\langle \phi X, Y\rangle = \langle AX, Y\rangle - H\langle X, Y\rangle.$$

It is easily checked that $\mathrm{tr}(\phi) = 0$ and that

$$|\phi|^2 = \frac{1}{2n}\sum (k_i - k_j)^2,$$

so that $|\phi|^2 = 0$ if and only if M^n is totally umbilical. Note that the eigenvalues of ϕ are given by $\mu_i = k_i - H$, so that $|\phi|^2 = \sum_i \mu_i^2 = |A|^2 - nH^2$ and

$$\sum_i k_i^3 = nH^3 + 3H\sum_i \mu_i^2 - \sum_i \mu_i^3. \tag{2}$$

We have the Gauss equation relating $\bar{R}$, H and $|A|^2$:

$$n(n-1)\bar{R} = n^2H^2 - |A|^2. \tag{3}$$

The standard examples of space-like umbilical hypersurfaces with constant mean curvature in the de Sitter space are given by

$$M^n = \{p \in S_1^{n+1}; \langle p, a\rangle = \tau\},$$

where $a \in \mathbf{R}_1^{n+2}$, $|a|^2 = \rho = 1, 0, -1$ and $\tau^2 > \rho$. The corresponding mean curvature H of such hypersurfaces satisfies

$$H^2 = \frac{\tau^2}{\tau^2 - \rho}$$

(see [12], for instance) and M^n is isometric to a hyperbolic space, an Euclidean space or a sphere if $\rho = 1, 0, -1$, respectively.

On the other hand, the hyperbolic cylinders are the hypersurfaces given by

$$M^n = \{p \in S_1^{n+1}; -p_0^2 + p_1^2 + \cdots + p_k^2 = -\sinh^2 r\}$$

with $r \in \mathbf{R}$ and $1 \leq k \leq n-1$. Such hyperbolic cylinders have constant mean curvature $nH = k \coth r + (n-k) \tanh r$. Thus we have

$$H^2 \geq \frac{4(n-1)}{n^2},$$

and the equality is attained for $k = 1$ and $\coth^2 r = (n-1)$. Their normalized scalar curvature is $R = 1 + (1/n)(2 + (n-2)\tanh^2 r)$. We point out that these examples have only two distinct constant principal curvatures at each point and one of them has multiplicity one. Moreover, they are isometric to the Riemannian product $H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$.

Also, we observe that the cylinders $H^{n-1}(1 - \coth^2 r) \times S^1(1 - \tanh^2 r)$ have normalized scalar curvature $R = 1 + (1/n)(2 + (n-2)\coth^2 r)$.

The Laplacian Δ acts on any C^2-function f defined on M as $(\Delta f)_{ij} = \Delta f_{ij} = \sum_k f_{ijkk}$ and the Laplacian of the second fundamental form h_{ij} is given by

$$\Delta h_{ij} = \sum_k h_{ijkk}.$$

We may write

$$\Delta h_{ij} = \sum_k (h_{ijkk} - h_{ikjk}) + \sum_k (h_{ikjk} - h_{ikkj}) + \sum_k (h_{ikkj} - h_{kkij}),$$

so that

$$\Delta h_{ij} = nH_{ij} + \sum_{m,k} h_{im} R_{mkjk} + \sum_{m,k} h_{km} R_{mijk},$$

where H_{ij} denotes the second covariant derivative of H.

Let $T = \sum_{i,j} T_{ij} \omega_i \otimes \omega_j$ be the symmetric tensor defined by

$$T_{ij} = nH\delta_{ij} - h_{ij}.$$

Following Cheng and Yau [6], we introduce the operator L_1 associated to T acting on C^2-functions f on M^n by

$$L_1 f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{4}$$

Around a given point $p \in M$ we choose an orthonormal frame field $\{e_1, \dots, e_n\}$ with dual frame field $\{w_1, \dots, w_n\}$ so that $h_{ij} = k_i \delta_{ij}$ at p. Using (3) and (4), we have

$$L_1(nH) = nH\Delta(nH) - \sum_i k_i (nH)_{ii} = \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i (nH)_{ii}$$
$$= \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta|A|^2 - n^2|\nabla H|^2 - \sum_i k_i (nH)_{ii}.$$

On the other hand, Simons formula (see [17, p. 320, (3.5)]) implies

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + n\sum_i k_i H_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2.$$

From the last two equations, we have

$$L_1(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla A|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{5}$$

3. Classification of complete constant scalar curvature hypersurfaces with $\bar{R} \leq \sup H^2$

Here we prove Theorem 1.1; for this we will need the following three lemmas.

Lemma 3.1. *Let M^n be an immersed space-like hypersurface in S_1^{n+1} with constant normalized scalar curvature. Then*

$$L_1(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + |\phi|^2(n - nH^2 + |\phi|^2) - nH\sum_i \mu_i^3. \tag{6}$$

Proof. Since R is constant, (3) and (5) imply

$$L_1(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j}(1 + k_i k_j)(k_i - k_j)^2 = |\nabla A|^2 - n^2|\nabla H|^2$$
$$+ \frac{1}{2}n\sum_i k_i^2 + \frac{1}{2}n\sum_j k_j^2 - \sum_{i,j} k_j k_i + \frac{1}{2}\sum_{i,j} k_i^3 k_j + \frac{1}{2}\sum_{i,j} k_i k_j^3 - \sum_{i,j} k_i^2 k_j^2.$$

Making $i = j$, we obtain

$$L_1(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + n|A|^2 - n^2H^2 + |A|^4 - nH\sum_i k_i^3.$$

Using (2) in the expression above, we have

$$L_1(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + n|\phi|^2 + |A|^4 - nH\sum_i \mu_i^3 - 3nH^2|\phi|^2 - n^2H^4.$$

Then

$$L_1(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + n|\phi|^2 + |\phi|^4 + n^2H^4$$
$$+ 2nH^2|\phi|^2 - 3nH^2|\phi|^2 - n^2H^4 - nH\sum_i \mu_i^3.$$

This yields

$$L_1(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + |\phi|^2(n - nH^2 + |\phi|^2) - nH\sum_i \mu_i^3,$$

which proves the lemma. □

Lemma 3.2 (Okumura [17, p. 210]). *Let μ_i, $i = 1, \dots, n$ be real numbers such that $\sum \mu_i = 0$ and $\sum \mu_i^2 = \beta \geq 0$, with β constant. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3,$$

and equality holds on the right-hand side, if and only if

$$\mu_1 = \cdots = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}}\beta, \qquad \mu_n = \sqrt{\frac{n-1}{n}}\beta.$$

Lemma 3.3. *Let M^n be an n-dimensional complete ($n \geq 3$) space-like hypersurface immersed into the de Sitter space S_1^{n+1} with constant normalized scalar curvature R and $\bar{R} = R - 1$. Then the following inequalities are equivalent*:
1. $|Z|^2 \leq n/((n-2)(n\bar{R}-2))[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n]$,
2. $(n-2)/n\sqrt{(n(n-1)\bar{R} + |Z|^2)(|Z|^2 - n\bar{R})} \leq n - 2(n-1)\bar{R} + (n-2)/n|Z|^2$,

where $|Z|^2 = \sup|A|^2$. Equivalence also holds in case of equality.

Proof. We will prove that (2) implies (1). Squaring (2) and simplifying, we get

$$\left(\frac{n-2}{n}\right)[(n-2)^2\bar{R} - 2(n - 2(n-1)\bar{R})]|Z|^2$$
$$\leq (n - 2(n-1)\bar{R})^2 + (n-2)^2(n-1)\bar{R}^2,$$

hence,

$$(n-2)(n\bar{R}-2)|Z|^2 \leq n[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n],$$

which gives (1). The other implication is immediate. □

We will also use the maximum principle at infinity for complete manifolds due to Omori [16] and Yau [19]:

Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2-function bounded from below on M^n. Then for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M$ such that

$$|\nabla f|(p_\varepsilon) < \varepsilon, \qquad \Delta f(p_\varepsilon) > -\varepsilon, \qquad \inf f \leq f(p_\varepsilon) < \inf f + \varepsilon.$$

Proof of Theorem 1.1. Applying Lemma 3.2 to the eigenvalues of ϕ, we obtain

$$\left|\sum \mu_i^3\right| \leq \frac{n-2}{\sqrt{n(n-1)}}(|\phi|^2)^{3/2}. \tag{7}$$

Using this fact and Lemma 3.1, we get

$$L_1(nH) \geq |\nabla A|^2 - n^2|\nabla H|^2 + |\phi|^2 P_H(|\phi|), \tag{8}$$

where P_H is the polynomial in $|\phi|$ given by

$$P_H(|\phi|) = n - nH^2 + |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|. \tag{9}$$

Since $\bar{R}$ is constant and positive, Lemma 4.1 of [2] implies

$$|\nabla A|^2 - n^2|\nabla H|^2 \geq 0.$$

Substituting this fact in (8), we obtain

$$L_1(nH) \geq |\phi|^2 P_H(|\phi|). \tag{10}$$

From (3),

$$|\phi|^2 = |A|^2 - nH^2 = \frac{n-1}{n}(|A|^2 - n\bar{R}). \tag{11}$$

Thus we may see $P_H(|\phi|)$ as $P_{\bar{R}}(|A|)$, where

$$\begin{aligned} P_{\bar{R}}(|A|) = {} & n - 2(n-1)\bar{R} + \frac{n-2}{n}|A|^2 \\ & - \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |A|^2)(|A|^2 - n\bar{R})}. \end{aligned} \tag{12}$$

Hence, we may write (10) as

$$L_1(nH) \geq \frac{n-1}{n}(|A|^2 - n\bar{R})P_{\bar{R}}(|A|). \tag{13}$$

We claim that $P_{\bar{R}}(\sqrt{\sup|A|^2}) \geq 0$. By hypothesis (see (1)),

$$\sup H^2 \leq C_n(\bar{R}) = \frac{1}{n^2}\left((n-1)^2\frac{n\bar{R}-2}{n-2} + 2(n-1) + \frac{n-2}{n\bar{R}-2}\right).$$

We use (3) to write this expression as

$$|Z|^2 \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n],$$

where again $|Z|^2 = \sup|A|^2$. By Lemma 3.3, this is equivalent to

$$\frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |Z|^2)(|Z|^2 - n\bar{R})} \leq n - 2(n-1)\bar{R} + \frac{n-2}{n}|Z|^2.$$

In view of (12), this last inequality shows that $P_{\bar{R}}(\sqrt{\sup|A|^2}) \geq 0$, so that our claim is proved.

On the other hand,

$$\begin{aligned} L_1(nH) &= \sum_{i,j}(nH\delta_{ij} - nh_{ij})(nH)_{ij} = \sum_i (nH - nh_{ii})(nH)_{ii} \\ &= n\sum_i H(nH)_{ii} - n\sum_i k_i(nH)_{ii} \leq n(\sup|H| - C)\Delta(nH), \end{aligned} \tag{14}$$

where $\sup|H|$ is the sup of the absolute value of the mean curvature H in M and $C = \min k_i$ the minimum of the principal curvatures in M.

From (13) and (14), we have

$$\frac{n-1}{n}(|A|^2 - n\bar{R})P_{\bar{R}}(|A|) \leq L_1(nH) \leq n(\sup|H| - C)\Delta(nH). \tag{15}$$

Since $\bar{R}$ is the second symmetric function of the eigenvalues of A, we have from the Newton inequalities that $\bar{R} \leq H^2$, and hence $\bar{R} \leq \sup H^2$. Gauss equation implies $Ric_M \geq (n-1) - \frac{1}{4}nH^2$, so that the Ricci curvature is bounded from below. Thus we may apply Omori and Yau's result to the function

$$f = \frac{1}{\sqrt{1+(nH)^2}},$$

which is a positive smooth function on M. Then,

$$|\nabla f|^2 = \frac{1}{4}\frac{|\nabla(nH)^2|^2}{(1+(nH))^2}, \tag{16}$$

$$\Delta f = -\frac{1}{2}\frac{\Delta(nH)}{(1+(nH))^{3/2}} + \frac{3|\nabla(nH)|^2}{4(1+nH)^{5/2}}. \tag{17}$$

Let $\{p_k\}$, $k \in N$, be a sequence of points in M given by Omori and Yau's result, such that

$$\lim_{k\to\infty} f(p_k) = \inf f, \qquad \Delta f(p_k) > -\frac{1}{k}, \qquad |\nabla f|^2(p_k) < \frac{1}{k^2}. \tag{18}$$

Using (16)–(18) and the fact that

$$\lim_{k\to\infty}(nH)(p_k) = \sup_{p\in M}(nH)(p),$$

we have

$$-\frac{1}{k} < -\frac{1}{2}\frac{\Delta(nH)}{(1+(nH))^{3/2}}(p_k) + \frac{3}{k^2}(1+nH(p_k))^{3/2}.$$

Hence,

$$\frac{\Delta(nH)}{(1+nH)^{3/2}}(p_k) < \frac{2}{k}\left(\frac{1}{\sqrt{1+(nH)(p_k)}} + \frac{3}{k}\right). \tag{19}$$

Evaluating (15) at the points p_k, making $k \to \infty$ and using (19), we have

$$\begin{aligned} 0 &\leq \frac{n-1}{n}(\sup|A|^2 - n\bar{R})P_{\bar{R}}\left(\sqrt{\sup|A|^2}\right) \leq L_1(n\sup|H|) \\ &\leq n(\sup|H| - C)\Delta(nH) < n(\sup|H| - C)(1+nH)^{3/2}(p_k)\frac{2}{k} \\ &\times\left(\frac{1}{\sqrt{1+(nH)(p_k)}} + \frac{3}{k}\right). \end{aligned}$$

If $k \to \infty$, we have that the last expression goes to zero, so either

$$(\sup|A|^2 - n\bar{R}) = 0, \quad \text{or} \quad P_{\bar{R}}\left(\sqrt{\sup|A|^2}\right) = 0.$$

In the first case, Gauss equation implies that

$$n\sup|\phi|^2 = (n-1)(\sup|A|^2 - n\bar{R}) = 0,$$

so that $|\phi|^2 = 0$ and M^n is totally umbilical.

In the second case, i.e., if $P_{\bar{R}}\left(\sqrt{\sup|A|^2}\right) = 0$, then Lemma 3.3 implies that

$$\sup|A|^2 = \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n]. \tag{20}$$

Using (11), we have

$$\sup H^2 = \frac{1}{n^2}\sup|A|^2 - \frac{n-1}{n}\bar{R}. \tag{21}$$

Substituting (20) in (21), we obtain $\sup H^2 = C_n(\bar{R})$ (defined in (1)) and so we have the following equality:

$$L_1(n\sup H) = \frac{n-1}{n}(\sup|A|^2 - n\bar{R})P_{\bar{R}}\left(\sqrt{\sup|A|^2}\right).$$

Thus, equality also holds on the right-hand side of Okumura's lemma (Lemma 3.2). After re-enumeration, we have $k_1 = k_2 = \cdots = k_{n-1}$, $k_1 \neq k_n$, where $k_1 = \tanh r$ and $k_n = \coth r$. Then M^n is isometric to $H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$, finishing the proof of the theorem. □

4. Examples of constant scalar curvature hypersurfaces satisfying $\sup H^2 > C_n(\bar{R})$

To show that the restriction $\sup H^2 \leq C_n(\bar{R})$ in Theorem 1.1 is sharp, we will give here for fixed $n \geq 3$, $\bar{R} > 2/n$ and every number $C > C_n(\bar{R})$, a complete hypersurface with constant scalar curvature $\bar{R}$ such that $\sup H^2 = C$. This is the content of Theorem 1.2 which we will prove here. First we recall the notion of a rotation hypersurface in the context of the de Sitter space [15] (see [7] for the Riemannian case).

Definition 4.1. Denote by P^k a k-dimensional linear subspace of R_1^{n+2} and by $\mathrm{O}(P^2)$ the identity component of the Lorentz group $\mathrm{O}(1, n+1)$ which leaves P^2 pointwise fixed. Choose P^2 and P^3 such that $P^2 \subset P^3$, and let C be a space-like curve in $S_1^{n+1} \cap (P^3 - P^2)$. Then the orbit of C under $\mathrm{O}(P^2)$ is a hypersurface $M^n \subset S_1^{n+1}$ called the rotation hypersurface generated by C, and C is called the profile curve of M. M is spherical (resp. parabolic, hyperbolic) if the restriction $\langle\,,\rangle|_{P^2}$ is a Lorentzian metric (resp. a Riemannian metric, a degenerate quadratic form).

In [7,15], do Carmo, Dajczer and Mori gave explicit parameterizations of these rotation hypersurfaces; for completeness, we write down these parameterizations in S_1^{n+1} for the spherical case and point out what happens in the other two cases.

Let $P^3 \subset S_1^{n+1}$ be the subspace $\{(y_0, y_1, y_2, 0, \dots, 0) \in \mathbf{R}_1^{n+2};\ y_i \in \mathbf{R}\}$, and the profile curve C be given by the functions $y_i = y_i(s)$, $i = 0, 1, 2$, parameterized by arc-length, so that the following conditions hold (from now on, we omit the variable s for convenience):

$$-y_0^2 + y_1^2 + y_2^2 = 1, \qquad -y_0'^2 + y_1'^2 + y_2'^2 = 1. \tag{22}$$

We may describe y_0, y_1 in terms of y_2, as follows. Set

$$y_0 = \sqrt{y_2^2 - 1}\cosh\varphi, \qquad y_1 = \sqrt{y_2^2 - 1}\sinh\varphi$$

for a function φ to be defined. Differentiating these equations and substituting into the second equation in (22), we get

$$1 = (y_2^2 - 1)\varphi'^2 - \frac{y_2^2 y_2'^2}{y_2^2 - 1} + y_2^2,$$

so that

$$\varphi = \int \frac{\sqrt{y_2'^2 + y_2^2 - 1}}{y_2^2 - 1}\,\mathrm{d}s.$$

Hence, the coordinates of the profile curve C may be given in terms of y_2 and its derivatives. In this setting, if we rotate C with respect to $P^2 = \{(y_0, y_1, 0, \dots, 0) \in \mathbf{R}_1^{n+2};\ y_i \in \mathbf{R}\}$, we may calculate the principal curvatures of our rotation hypersurface (see [7,15]), which are given by

$$\kappa_i = \frac{\sqrt{y_2'^2 + y_2^2 - \delta}}{y_2}, \tag{23}$$

$$\kappa_n = \frac{y_2'' + y_2}{\sqrt{y_2'^2 + y_2^2 - \delta}}, \tag{24}$$

where $i = 1, \dots, n-1$ and $\delta = -1, 0, 1$, when the hypersurface is spherical, parabolic or hyperbolic, respectively.

It is worth noting that the hyperbolic cylinders $H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$, $r > 0$ already mentioned in this paper may be given as rotation hypersurfaces as follows: if $2/n < \bar{R} < 1$, we set $y_2 = \cosh r$, where

$$\tanh^2 r = \frac{n\bar{R} - 2}{n - 2},$$

while if $\bar{R} > 1$, we use $y_2 = \sinh r$, where

$$\coth^2 r = \frac{n\bar{R} - 2}{n - 2}.$$

Returning to a general rotation hypersurface, we may use the expressions for the principal curvatures to obtain the following formulas for H and $\bar{R}$:

$$nH = (n-1)\frac{\sqrt{y_2'^2 + y_2^2 - \delta}}{y_2} + \frac{y_2'' + y_2}{\sqrt{y_2'^2 + y_2^2 - \delta}}, \tag{25}$$

$$n\bar{R} = (n-2)\frac{y_2'^2 + y_2^2 - \delta}{y_2^2} + 2\frac{y_2'' + y_2}{y_2}. \tag{26}$$

Now we use the following result (see [9,18]):

If $\bar{R}$ is constant, the last equation has the first integral given by

$$G(y_2, y_2') = y_2^{n-2}(y_2'^2 + y_2^2 - 1 - \bar{R}y_2^2). \tag{27}$$

We translate the study of the level curves of G in the (y_2, y_2')-plane into information about the profile curves of our rotation hypersurfaces; for example, the critical points $(y_c, 0)$ of G, satisfying

$$n(1-\bar{R})y_c^2 - (n-2)\delta = 0,$$

or, equivalently,

$$y_c^2 = \frac{(n-2)\delta}{n(1-\bar{R})}, \tag{28}$$

correspond exactly to the hyperbolic cylinders

$$H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$$

with principal curvatures $k_i = \tanh r$ and $k_n = \coth r$. We use (25), (26) and (28) to write H^2 in terms of $\bar{R}$, obtaining

$$H^2 = \frac{1}{n^2}\left((n-1)^2\frac{n\bar{R}-2}{n-2} + 2(n-1) + \frac{n-2}{n\bar{R}-2}\right),$$

which is the value of $C_n(\bar{R})$ defined in (1) and corresponding to the hyperbolic cylinders, thus defining a function $\bar{R} \mapsto H^2$. Fig. 1 (see Introduction) shows the graph of this function on a $(\bar{R}, \sup H^2)$-plane. We remark that in the vertical axis we plot $\sup H^2$ and not H^2. Also, if we take an umbilical hypersurface, we have $\sup H^2 = H^2 = \bar{R}$, so that these hypersurfaces are represented by the first quadrant bisector ray.

Proof of Theorem 1.2. Regarding Fig. 1, we will show that every point above the graph of $C_n(\bar{R})$ has associated a rotation space-like hypersurface such that $(\bar{R}, \sup H^2)$ are the coordinates of the given point. As before, we work out the details only in the spherical case.

The level curves of the first integral given in (27) give rise to space-like hypersurfaces with constant scalar curvature. In particular, we obtain complete space-like hypersurfaces whenever the level curve stays inside the region $\{y_2 > 1\}$. If $2/n < \bar{R} < 1$, the critical point $(y_c, 0)$ mentioned above is such that $y_c > 1$ (this is the reason to consider here $\bar{R} > 2/n$; in case $0 < \bar{R} \leq 2/n$ we obtain only non-complete examples). An easy calculation shows that this critical point $(y_c, 0)$ is a minimum, so that the level curves near this point are closed.

These nearby closed level curves have its y_2-coordinate bounded between, say, y_* and y^* (these values, of course, depend on the level curve we are considering) and so the profile curve is contained in a strip given by $y_* \leq y_2 \leq y^*$, containing the critical point y_c.

We claim that, if we consider all closed level curves which stay inside the region $\{y_2 > 1\}$, then, for the corresponding *complete* space-like hypersurfaces (with $\bar{R}$ fixed), the value of $\sup H^2$ varies in $[C_n(\bar{R}), \infty)$.

To prove the claim, we first write H^2 in terms of $\bar{R}$, y_2 and y_2'; from (26),

$$\frac{y_2'' + y_2^2}{\sqrt{y_2'^2 + y_2^2 - \delta}} = \frac{y_2}{2\sqrt{y_2'^2 + y_2^2 - \delta}} \left(n\bar{R} - (n-2)\frac{y_2'^2 + y_2^2 - \delta}{y_2^2} \right)$$
$$= \frac{n\bar{R}}{2} \frac{y_2}{\sqrt{y_2'^2 + y_2^2 - \delta}} - (n-2)\frac{\sqrt{y_2'^2 + y_2^2 - \delta}}{y_2},$$

so that

$$H = \frac{1}{2}\left(\frac{\sqrt{y_2'^2 + y_2^2 - \delta}}{y_2} + \bar{R}\frac{y_2}{\sqrt{y_2'^2 + y_2^2 - \delta}} \right). \tag{29}$$

Fixing the value of the first integral $G(y_2, y_2') = k$, so that

$$y_2^{n-2}(y_2'^2 + y_2^2 - \delta - \bar{R}y_2^2) = k,$$

we get

$$\frac{\sqrt{y_2'^2 + y_2^2 - \delta}}{y_2} = \sqrt{\frac{\bar{R}y_2^n + k}{y_2^n}}.$$

Using this last expression in (29), we get

$$H = \frac{1}{2}\left(\sqrt{\frac{\bar{R}y_2^n + k}{y_2^n}} + \bar{R}\sqrt{\frac{y_2^n}{\bar{R}y_2^n + k}} \right).$$

It is easy to show that H, as a function of y_2, is differentiable, defined in an interval of the form $(1, y_{\bar{R}}]$, where

$$G(y_{\bar{R}}, 0) = G(1, 0) = -\bar{R},$$

and that H is *decreasing* in its domain. These properties imply that for a closed level curve $G(y_2, y_2') = k$ of G, with y_2-extreme values y_* and y^*,

$$\sup H^2 = H^2(y_*),$$

so that $\sup H^2$ is a differentiable function of y_*.

If we make $y_* \to 1^+$, the value k of G tends to $G(1,0) = -\bar{R}$ and we have

$$\lim_{y_* \to 1^+} \sup H^2 = \lim_{y_* \to 1^+} \frac{1}{4}\left(\sqrt{\frac{\bar{R}y_*^n + k}{y_*^n}} + \bar{R}\sqrt{\frac{y_*^n}{\bar{R}y_*^n + k}}\right)^2 = \infty.$$

On the other hand,

$$\lim_{y_* \to y_c^-} \sup H^2 = \lim_{y_* \to y_c^-} H^2(y_*) = H^2(y_c).$$

But as y_c corresponds to a hyperbolic cylinder, $H^2(y_c) = C_n(\bar{R})$, and so the claim and the theorem are proved. □

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